The Effect of Shallow Water on Wave Resistance.

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1. The general character of experimental results dealing with the effect of shallow water on ship resistance may be stated briefly as follows:—At low velocities the resistance in shallow water is greater than in deep water, the speed at which the excess is first appreciable varying with the type of vessel. As the speed increases, the excess resistance increases up to a maximum at a certain critical velocity, and then diminishes. With still further increase of speed, the resistance in shallow water ultimately becomes, and remains, less than that in deep water at the same speed. The maximum effect is the more pronounced the shallower the water. For further details and references one may refer to standard treatises, but one quotation may be made in regard to the critical velocity: "This maximum appears to be at about a speed such that a trochoidal wave travelling at this speed in water of the same depth is about $1\frac{1}{4}$ times as long as the vessel. . . . It was at one time supposed that the speed for maximum increase in resistance was that of the wave of translation. This, however, holds only for water whose depth is less than 0·2 times the length of the vessel. For greater depths the speed of the wave of translation rapidly becomes greater than the speed of maximum increase of resistance."* In a recent analysis of the data, H. M. Weitbrecht† expresses a similar conclusion by stating that for each depth of water there is a critical velocity, but that the critical velocity does not vary as the square root of the corresponding depth.

It should be noted that experimental results are for the total resistance. If we assume that this can be separated into three terms, which are simply additive, namely, eddy, frictional, and wave-making resistance, it must be admitted that probably all are affected by limited depth of water. However, the main differences are due to the altered wave-making, and the general explanation is to be found in the fact that there is a limiting velocity, $\sqrt{(gh)}$, for simple straight-crested waves on water of depth h.

Leaving aside the difficult problem of a solid body towed or driven through the water, we may study the allied problem of a given distribution of surface pressure and the associated wave resistance. Previous calculations of wave resistance have been limited to a line distribution of pressure, involving therefore, only straight-crested parallel waves and so emphasising the connection between the critical velocity and that of the wave of translation. In the present paper I obtain an expression for the wave resistance of a surface pressure symmetrical about a point, and moving over water of finite depth. The result is in the form of a definite integral, which has been evaluated by numerical and graphical methods so as to give graphs of the variation of wave resistance with speed for different values of the ratio of the depth of water to the length associated with the pressure distribution. The graphs are of special interest in the cases intermediate between the two extremes of deep water and shallow water. They show the double effect of limited depth, in lowering the normal wave-making speed of the ship and in increasing the magnitude of the effect as the speed approaches that of the wave of translation. The results are discussed in their bearing upon the experimental results which have just been described.

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* D. W. Taylor, 'Speed and Power of Ships,' p. 114; also G. S. Baker, 'Ship Form, Resistance and Screw Propulsion,' p. 134.

† H. M. Weitbrecht, 'Jahrbuch d. Schiffbautech. Gesell.,' vol. 22, p. 122 (1921).

2. In a previous paper* I worked out the case of a symmetrical surface pressure moving over deep water. The present analysis is on exactly similar lines, except for suitable changes in the expressions; it may be sufficient, therefore, to set forth the calculation briefly, referring to the previous paper for further detail in the argument.

Take axes Ox, Oy in the undisturbed horizontal surface of water of depth h and Oz vertically upwards. For an initial impulse symmetrical about the origin, that is if the initial data are

$$\rho\phi_0 = \mathrm{F}(\varpi), \qquad \zeta = 0, \tag{1}$$

where $\varpi^2 = x^2+y^2$, the velocity potential and surface elevation in the subsequent fluid motion are given by

$$\rho\phi = \int_0^\infty f(\kappa)\cosh\kappa(z+h)\,\mathrm{sech}\,\kappa h\,\mathrm{J}_0(\kappa\varpi)\cos(\kappa\mathrm{V}t)\,\kappa\,d\kappa,$$

$$g\rho\zeta = -\int_0^\infty f(\kappa)\,\mathrm{J}_0(\kappa\varpi)\sin(\kappa\mathrm{V}t)\,\kappa^2\mathrm{V}\,d\kappa, \tag{2}$$

where $$\mathrm{V}^2 = (g/\kappa)\tanh\kappa h,$$

$$f(\kappa) = \int_0^\infty \mathrm{F}(\alpha)\,\mathrm{J}_0(\kappa\alpha)\,\alpha\,d\alpha. \tag{3}$$

We obtain the effect of a travelling pressure system by integrating with respect to the time. We shall suppose that the system has been moving for a long time with uniform velocity, c, in the direction of Ox. Transferring to a moving origin at the centre of the system, we replace x in (2) by $x+ct$, and we find for the surface elevation

$$g\rho\zeta = -\int_0^\infty e^{-1/2\mu t}\,dt\int_0^\infty f(\kappa)\,\mathrm{J}_0[\kappa\{(x+ct)^2+y^2\}^{1/2}]\sin(\kappa\mathrm{V}t)\,\kappa^2\,\mathrm{V}\,d\kappa, \tag{4}$$

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* 'Roy. Soc. Proc.,' A, vol. 95, p. 354 (1918).

where $f(\kappa)$ is found from the assigned pressure distribution, $p = \mathrm{F}(\varpi)$, by means of (3). The factor exp. $(-\frac{1}{2}\mu t)$ serves to keep the integrals determinate, so that they give a solution which corresponds to the main part of the surface waves trailing aft from the moving disturbance. It is to be noted that ultimately μ is made zero in the final results, and it is only retained in the intermediate analysis to a degree sufficient to attain its chief purpose. It should be stated also that all the analysis is subject to the usual limitation that the slope of the surface is supposed to be always small.

We take the wave resistance to be the resolved part of the pressure system in the direction of motion, or

$$\mathrm{R} = \int \mathrm{F}(\varpi) \frac{\partial \zeta}{\partial x} d\mathrm{S}, \tag{5}$$

taken over the whole surface.

The disturbance (4) may be analysed into plane waves ranged at all possible angles to Ox. Substituting

$$\pi \mathrm{J}_0 \left[\kappa \{ (x+ct)^2 + y^2 \}^{1/2}\right] = \int_0^\pi e^{i\kappa(x+ct)\cos\phi} \cos(\kappa y \sin\phi)\, d\phi, \tag{6}$$

we can integrate with respect to t, and obtain, after rejecting superfluous terms in μ,

$$\int_0^\infty e^{-1/2\mu t} e^{i\kappa ct\cos\phi} \sin(\kappa \mathrm{V} t)\, dt = -\frac{\mathrm{V}\sec^2\phi}{\kappa c^2 - g\sec^2\phi\tanh\kappa h + i\mu c\sec\phi}. \tag{7}$$

Using this in (4), the surface elevation can be expressed in the form

$$2\pi\rho\zeta = \int_{-\pi/2}^{\pi/2} \sec^2\phi\, d\phi \int_0^\infty \kappa f(\kappa) \tanh\kappa h \left\{ \frac{e^{i\kappa(x\cos\phi + y\sin\phi)}}{\kappa c^2 - g\sec^2\phi\tanh\kappa h + i\mu c\sec\phi} + \frac{e^{-i\kappa(x\cos\phi+y\sin\phi)}}{\kappa c^2 - g\sec^2\phi\tanh\kappa h - i\mu c\sec\phi} \right\} d\kappa. \tag{8}$$

3. We simplify the calculations by specifying the surface distribution of pressure as

$$p = \mathrm{F}(\varpi) = \mathrm{A}l/(l^2 + \varpi^2)^{3/2}, \tag{9}$$

where A and l are constants. It follows from (3) that $f(\kappa) = \mathrm{A}e^{-\kappa l}$. Now in (8) consider an element making an angle ϕ with the axis Ox. Change to axes Ox', Oy', given by $x' = x\cos\phi + y\sin\phi$, $y' = y\cos\phi - x\sin\phi$. Then the integral with respect to κ becomes

$$\int_0^\infty \kappa e^{-\kappa l} \tanh\kappa h \left\{ \frac{e^{i\kappa x'}}{\kappa c^2 - g\sec^2\phi\tanh\kappa h + i\mu c\sec\phi} + \frac{e^{-i\kappa x'}}{\kappa c^2 - g\sec^2\phi\tanh\kappa h - i\mu c\sec\phi} \right\} d\kappa. \tag{10}$$

As in similar plane wave problems, this integral can be modified by integrating round a suitable contour in the plane of a complex variable; the expressions then divide into two types according as the integrand has or has not a pole within the contour. The surface disturbance corresponding to (10) is seen then to consist, in general, of a surface elevation symmetrical with respect to the line $x\cos\phi + y\sin\phi = 0$, together with a regular train of waves in the rear of this line; but the latter part only occurs if $c^2\cos^2\phi < gh$. In evaluating the wave resistance by (5) for the symmetrical distribution (9), we see that we need only consider the regular train of waves. By calculating the residue of the integrand in (10), collecting the results and finally making μ zero, we find that the regular waves, when they occur, are given by

$$\frac{4\pi A c^2 \kappa^2 e^{-\kappa l}\sin(\kappa x')}{g\sec^2\phi\,(c^2 - gh\sec^2\phi) + \kappa^2 c^4 h}, \tag{11}$$

where κ is the root of

$$\kappa c^2 - g\sec^2\phi\tanh\kappa h = 0\,; \qquad gh\sec^2\phi > c^2. \tag{12}$$

From (5) and (11), the contribution of this element to the wave resistance is

$$\frac{4\pi A c^2 \kappa^3 e^{-\kappa l}\cos\phi}{g\sec^2\phi\,(c^2 - gh\sec^2\phi) + \kappa^2 c^4 h}\int_{-\infty}^{\infty} dy' \int_{-\infty}^{0} \frac{Al\cos(\kappa x')}{(x'^2 + y'^2 + l^2)^{3/2}}\,dx'$$

$$= \frac{4\pi^2 A^2 c^2 \kappa^3 e^{-2\kappa l}\cos\phi}{g\sec^2\phi\,(c^2 - gh\sec^2\phi) + \kappa^2 c^4 h}. \tag{13}$$

Summing for the different elements, from (13) and (7), we have finally for the wave resistance

$$R = \frac{4\pi A^2 c^2}{\rho}\int_{\phi_0}^{\pi/2} \frac{\kappa^3 e^{-2\kappa l}\sec\phi\,d\phi}{g\sec^2\phi\,(c^2 - gh\sec^2\phi) + \kappa^2 c^4 h}, \tag{14}$$

where κ satisfies $\kappa c^2 = g\sec^2\phi\tanh\kappa h$, and the lower limit ϕ_0 is given by

$$\phi_0 = 0, \text{ for } c^2 < gh\,; \qquad \phi_0 = \text{arc}\cos(gh/c^2)^{1/2}, \text{ for } c^2 > gh. \tag{15}$$

4. We may notice, in the first place, that (14) reduces to the expression given previously for deep water; making $h \to \infty$, we find

$$R = (4\pi g^2 A^2/\rho c^6)\int_0^{\pi/2} \sec^5\phi\, e^{-2(gl/c^2)\sec^2\phi}\,d\phi$$

$$= \frac{\pi^2 A^2 x^3 e^{-x}}{2g\rho l^3}\left\{ i H_0^{(1)}(ix) - \frac{1+2x}{2x} H_1^{(1)}(ix)\right\}, \tag{16}$$

where $x = gl/c^2$, and the result is expressed in terms of Bessel functions of which Tables are available. For finite values of the ratio h/l, the value of R for given values of c can only be obtained from (14) by numerical and graphical methods. After some preliminary trial, the following plan was adopted. With $p = h/l$, and $\alpha = \kappa h$, using the relation between κ and ϕ, (14) can be put in the form

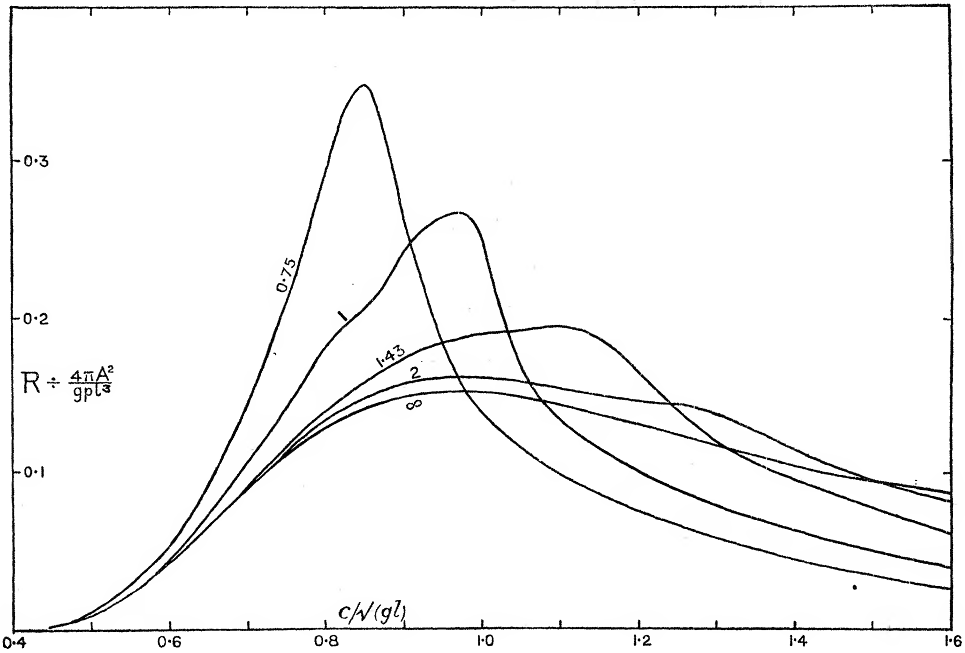
$$R = \frac{4\pi A^2 x^{1/2}}{g\rho l^3 p^{5/2}} \int_{\phi_0}^{\pi/2} \frac{\alpha^{7/2} e^{-2\alpha/p} \coth^{1/2}\alpha}{\alpha^2 + \alpha\coth\alpha - \alpha^2\coth^2\alpha}\, d\phi, \tag{17}$$

with

$$\alpha\coth\alpha = px\sec^2\phi. \tag{18}$$

For a given value of p, the integrand of (17), which we may denote by $f(\alpha)$, was calculated for values of α ranging from zero to 3 at intervals of 0·2, and in certain cases also at unit intervals up to the value 10. Taking next an assigned value of x, the value of ϕ corresponding to each value of α was found from (18). The integrand $f(\alpha)$ was then graphed on a base of ϕ, giving a curve for each value of x; the area of the curve was taken by an Amsler radial planimeter, and then the value of (17) was obtained. The calculations are rather lengthy and it is unnecessary to repeat them here.

The process was carried out for $p = 2$, 1·43, 1, 0·75, with about a dozen values of x in each case; some estimates were also made for $p = 0·5$, to confirm the general deductions. Further, the values for $p = \infty$ were calculated from (16). The results are shown in the figure, where the unit for R is $4\pi A^2/g\rho l^3$, and for c is $\sqrt{(gl)}$.



5. The curve for deep water, $p = \infty$, has a single maximum at a velocity slightly less than $\sqrt{(gl)}$. At this velocity the corresponding length of

simple transverse waves is about $2\pi l$; this may be called the principal wave-making length of the disturbance, to use a term from the theory of ship resistance. Taking next the curve for $p = 2$, we can see indications of two maxima. The first occurs at about the point 0·97 on the velocity scale; it clearly corresponds to the deep water maximum, and comes lower down the scale, because waves of given length occur at a lower velocity as the depth diminishes. There is also a second maximum at a velocity of about 1·25; this is due to the other factor in the resistance, namely, the increased effect as the velocity approaches the velocity $\sqrt{(gh)}$ of the so-called wave of translation, which in this case is at the point 1·41 on the velocity scale.

From the next curves, $p = 1·43$ and $p = 1$, we see the increasing importance of the latter effect as the depth becomes less. For the curve $p = 1·43$, there is a maximum near the velocity 1·1, the corresponding value of $\sqrt{(gh)}$ on the scale being 1·2. There is no other actual maximum, but there is an enhanced resistance at about 0·92, followed by a flattening of the curve between that point and the point 1·05; we may take the increased effect at 0·92 to correspond to the deep water maximum in the lower curves. Similarly for the curve $p = 1$, the corresponding values are: increased effect at about 0·81, diminished slope of curve between 0·82 and 0·9, maximum at 0·97, velocity of wave of translation 1·0. The last curve, $p = 0·75$, shows that, as the depth becomes small, the second effect becomes the predominant feature; the excess resistance increases rapidly in magnitude, and occurs practically at the velocity $\sqrt{(gh)}$. This effect is still more pronounced for $p = 0·5$, but the results are not shown in the figure. It is obvious that, as the ratio of h/l diminishes, the disturbance becomes more like that due to a line disturbance; in simple calculations on the latter assumption, the resistance increases indefinitely at the velocity $\sqrt{(gh)}$, and falls suddenly to zero above that velocity. It will be seen from the figure that in all cases the resistance falls after the velocity $\sqrt{(gh)}$, as, in fact, may be deduced directly from the expression (17).

In a comparison between these results and the experimental curves of ship resistance described in § 1, it is advisable to consider in each case the difference between the resistance in water of a given depth and that in deep water; in this sense the results agree in character. Thus the first effect of finite depth may be regarded as due to the lowering of the chief wave-making velocity; it is only when the depth of water becomes of the same order as the beam of the ship that the critical velocity is practically that of the wave of translation.

In describing the experimental curves, it was stated that the excess resistance has a maximum value at a certain critical velocity. But there is one exceptional set of curves, obtained at the United States Model Basin,* which shows two maxima: a phenomenon which has not received explanation. It is conceivable that this may be a case in which the two maxima indicated in the intermediate curves of the present paper have become prominent through some unusual features of the model. In this connection, it must be remembered that the present calculations are based upon a surface pressure of specially simple type, one symmetrical round a point; one could extend the calculations by integration, as in the previous results for deep water, so as to apply to a pressure distribution, giving a better analogy with ship form. It may be anticipated that the results would be of the same character in general, though no doubt better agreement could be obtained in certain details.

The Diffraction of Plane Electromagnetic Waves by a Perfectly Reflecting Sphere.

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Introduction.—The problem of the diffraction of electromagnetic waves by a perfectly conducting sphere is of interest both from the point of view of wireless and from that of physical optics. Two cases may be considered: (1) when the source of the waves is a Hertzian oscillator on the surface of the sphere; and (2) when the waves are plane. The formal series solution of both these problems has been given by several writers, including Sir J. J. Thomson, the late Lord Rayleigh, and Prof. H. M. Macdonald. For a sphere of which the radius is small compared with the wave-length the series converge rapidly and are suitable for computation, but for a large sphere the important terms are far on in the series and the latter must be transformed in order to get formulæ which may be of use.

For case (1) this problem has been attacked by Macdonald, Poincaré, J. W. Nicholson, A. E. H. Love and several others, but case (2) has not attracted nearly so much attention. This paper, then, deals with case (2).

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* D. W. Taylor, *loc. cit.*, p. 115.